

# Non-abelian pseudomeasures and congruences between abelian Iwasawa $L$ -functions

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Dedicated to Professor J.-P. Serre on his 80<sup>th</sup> birthday

**ABSTRACT.** The paper starts out from pseudomeasures (in the sense of Serre) which hold the arithmetic properties of the abelian  $l$ -adic Artin  $L$ -functions over totally real number fields. In order to generalize to non-abelian  $l$ -adic  $L$ -functions, these abelian pseudomeasures must satisfy congruences which are introduced but not yet known to be true. The relation to the “equivariant main conjecture” of Iwasawa theory is discussed.

Fix an odd prime number  $l$  and a finite field extension  $k/\mathbb{Q}$  with  $k$  totally real. Let  $k_\infty$  be the cyclotomic  $\mathbb{Z}_l$ -extension of  $k$  and  $K \supset k_\infty$  be a totally real Galois extension of  $k$  with Galois group  $G$  and so that  $[K : k_\infty]$  is finite. Setting  $H = G_{K/k_\infty}$  and  $\Gamma_k = G_{k_\infty/k}$ , we get the group extension  $1 \rightarrow H \rightarrow G \rightarrow \Gamma_k \rightarrow 1$ .

Let first  $G$  be abelian. We consider the group algebra  $\mathcal{Q}G = \text{Quot}(\Lambda\Gamma)[H]$  which results from a splitting of the above group extension, with  $\Lambda\Gamma (\simeq \mathbb{Z}_l[[T]])$  the Iwasawa algebra of a preimage  $\Gamma$  of  $\Gamma_k$  in  $G$ . From Serre’s interpretation [Se] of the work [DR] of Deligne and Ribet on abelian  $L$ -functions over totally real fields it follows that there is a unique element  $\lambda_{K/k} \in \mathcal{Q}G$  that encodes all the  $l$ -adic  $L$ -functions  $L_l(s, \chi)$  of  $k_\infty/k$  for the characters  $\chi$  of  $G$  with open kernel. In fact, choosing a preimage  $\gamma \in \Gamma$  of a generator  $\gamma_k \in \Gamma_k$  and identifying  $\gamma - 1$  with the variable  $T$  (in  $\Lambda\Gamma \simeq \mathbb{Z}_l[[T]]$ ), this *pseudomeasure*  $\lambda_{K/k}$  admits a power series expansion

$$\lambda_{K/k} = \sum_{m \geq -1} a_m T^m \text{ with } a_m \in \mathbb{Z}_l[H]$$

which has the property that

$$\sum_{m \geq -1} \chi(a_m) (\chi(\gamma) u^n - 1)^m = L_l(1 - n, \chi) \text{ for } n \geq 1 \text{ and all } \chi ,$$

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where  $u \in 1 + l\mathbb{Z}_l$  is determined by the action of  $\gamma_k$  on the  $l$ -power roots of unity. In particular,  $\lambda_{K/k} \in (\Lambda_\bullet \Gamma)[H]$ , with  $\bullet$  denoting localization <sup>1</sup> at the prime ideal  $l\Lambda\Gamma$  of  $\Lambda\Gamma$ ; moreover,  $\lambda_{K/k}$  is a unit in  $\mathcal{Q}G$ . If Iwasawa's  $\mu$ -invariant of  $K/k$  vanishes, then <sup>2</sup>  $\lambda_{K/k}$  is even a unit in  $\Lambda_\bullet G$ .

Though the pseudomeasure carries all the information about the abelian  $l$ -adic  $L$ -functions, its arithmetic properties are not well understood at present as far as we know.

We now drop the assumption that  $G$  is abelian. Then the total ring  $\mathcal{Q}G$  of fractions <sup>3</sup> of the Iwasawa algebra  $\Lambda G = \mathbb{Z}_l[[G]]$  of  $G$  is a finite dimensional semisimple algebra over  $\mathcal{Q}\Gamma$ , for every central open subgroup  $\Gamma$  of  $G$  which is isomorphic to  $\mathbb{Z}_l$ . Moreover, there is a homomorphism

$$\text{Det} : K_1(\mathcal{Q}G) \rightarrow \text{Hom}(R_l(G), (\mathcal{Q}^c \Gamma_k)^\times)$$

taking  $x \in K_1(\mathcal{Q}G)$  to the map  $\text{Det } x$  which assigns an element in  $(\mathcal{Q}^c \Gamma_k)^\times \stackrel{\text{def}}{=} (\mathbb{Q}_l^c \otimes_{\mathbb{Q}_l} \mathcal{Q} \Gamma_k)^\times$  to each character  $\chi$  of  $G$  (with open kernel and with values in a fixed algebraic closure  $\mathbb{Q}_l^c$  of  $\mathbb{Q}_l$ ). Above,  $R_l(G)$  is the  $\mathbb{Z}$ -span of the  $\mathbb{Q}_l^c$ -irreducible characters  $\chi$  with open kernel (see also [RW2, p.588]). Given a finite set  $S$  of primes of  $k$  which contains all primes above  $\infty$  and  $l$ , Greenberg [Gr] has generalized  $l$ -adic  $L$ -functions  $L_{l,S}(s, \chi)$  to non-abelian characters  $\chi$  of  $G$  (with open kernel), and these in turn define the Iwasawa  $L$ -function  $\chi \mapsto L_{K/k}(\chi)$  (see [RW2, p.563]) which belongs to  $\text{Hom}(R_l(G), (\mathcal{Q}^c \Gamma_k)^\times)$ .

The natural question arises whether  $L_{K/k}$  has a preimage in  $K_1(\mathcal{Q}G)$ . Any such <sup>4</sup> may be regarded as a non-abelian analogue of Serre's abelian pseudomeasure. If  $G$  is abelian,  $K_1(\mathcal{Q}G) = (\mathcal{Q}G)^\times$  and we are back in Serre's situation. If Iwasawa's  $\mu$ -invariant of  $K/k$  vanishes, the abelian case hints at finding a  $\lambda$  in  $K_1(\Lambda_\bullet G)$  <sup>5</sup> with  $\text{Det } \lambda = L_{K/k}$ ; its image in  $K_1(\mathcal{Q}G)$  is then a non-abelian pseudomeasure for  $K/k$ .

The purpose of this paper is to formulate conditions which guarantee the existence of non-abelian pseudomeasures in the case when  $\mu(K/k) = 0$ . These are stated in terms of congruences between values of Iwasawa  $L$ -functions and lead to hypothetical new congruences between abelian pseudomeasures.

Here is a short outline of the contents of the paper. In §1 we review results from [RW2,3] and also provide some notation that is used throughout. In this and in the later sections we restrict ourselves to pro- $l$  groups  $G$ , though this would not be necessary. We also assume that Iwasawa's  $\mu$ -invariant of  $K/k$  vanishes (it then does so for all intermediate extensions  $\tilde{K}/k$  and  $K/k'$ , with  $\tilde{K}$  and  $k'$  the fixed fields of a finite normal subgroup  $N \triangleleft G$  respectively an open subgroup  $G' \leq G$ , see [RW3, footnote 1]). The next two sections introduce two kinds of congruences between Iwasawa  $L$ -functions, which hold precisely when  $L_{K/k} \in \text{Det } K_1(\Lambda_\bullet G)$ .

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<sup>1</sup>i.e., inverting all elements in  $\Lambda\Gamma \setminus l \cdot \Lambda\Gamma$

<sup>2</sup>See [RW3, proof of Corollary to Theorem 9]. The  $\mu$ -invariant is that of the Galois group  $X = G(M/K)$  of the maximal abelian  $l$ -extension  $M$  of  $K$  unramified outside  $l$ .

<sup>3</sup>we invert all central regular elements of  $\Lambda G$

<sup>4</sup>which should be unique according to a conjecture of Suslin (see [RW2, p.565, Remark (E)])

<sup>5</sup>where  $\Lambda_\bullet G$  is obtained by inverting all central elements of  $\Lambda G$  which are regular in  $\Lambda G/l\Lambda G$

The second kind is reformulated in terms of congruences between abelian pseudomeasures. Section 4 discusses the first kind for special pro- $l$  groups  $G$  (those with an abelian subgroup of index  $l$ ) and reduces them to the above mentioned congruences between abelian pseudomeasures if these groups  $G$  have nilpotency class 2.

Finally, in an appendix, we prove that the existence of a pseudomeasure in  $K_1(\mathcal{Q}G)$  does not depend on the size of  $S$  as long as  $S$  contains all primes dividing  $l$  or  $\infty$ . The appendix includes moreover a short review of the “equivariant main conjecture” of Iwasawa theory, as introduced in [RW2], and its equivalence to the existence of non-abelian pseudomeasures in  $K_1(\mathcal{Q}G)$ . Here, the set  $S$  must be *sufficiently large*, i.e., contain all infinite primes and all primes of  $k$  whose ramification index in  $K/k$  is divisible by  $l$ <sup>6</sup>. As a corollary, the “equivariant main conjecture” is independent of the choice of  $S$ .

**Added in proof:** From the referee’s report we have learned of the manuscript *Iwasawa theory of totally real fields for Galois extensions of Heisenberg type* by K. Kato, which is closely related to this paper. We would like to thank Professor Kato for sending us the “Very preliminary version” of January 15, 2007.

In this manuscript, Kato sketches a proof of the main conjecture of its title, which (in Lie dimension 1) is equivalent to the “equivariant main conjecture” (see §5), in the special case of Galois groups of Heisenberg type. Very roughly, the idea is to identify certain congruences between abelian pseudomeasures (we would say), and then to verify them by using methods from [DR].

When the Heisenberg situation overlaps with §4, these certain congruences are a variation of the single hypothetical congruence  $\lambda_{K/k'} \equiv \text{ver}(\lambda_{K_{\text{ab}}/k}) \pmod{\mathcal{T}'}$  which is equivalent to the main conjecture by Propositions 3.2 and 4.4. So long as the relevant congruences can be explicitly stated, there is cause for optimism that Kato’s ideas will permit a proof of the main conjecture. In this connection we would like to mention that the nilpotency class 2 assumption in Proposition 4.4 has meanwhile been eliminated.

## 1. NOTATION AND BASIC RESULTS

Whenever we write  $\Gamma$  we mean an open subgroup of  $G$  which is isomorphic to  $\mathbb{Z}_l$ ; whenever we say “character” we mean a  $\mathbb{Q}_l^c$ -valued character of  $G$  with open kernel. Such characters are said to be of type W, and always denoted by  $\rho$ , if  $\rho$  is  $\mathbb{Q}_l^c$ -irreducible and satisfies  $\rho(h) = 1$  for  $h \in H$ . Each  $\rho$  determines the automorphism  $\rho^\sharp$  of the field  $\mathcal{Q}^c\Gamma_k$ , induced by  $\rho^\sharp(\gamma) = \rho(\gamma)\gamma$  for  $\gamma \in \Gamma_k$ . The  $l$ -th Adams operation on  $R_l(G)$  is denoted by  $\psi_l$ , so  $(\psi_l\chi)(g) = \chi(g^l)$  for  $g \in G$ . And  $\Psi$  is the endomorphism of  $\Lambda\Gamma$  induced by  $\Psi(\gamma) = \gamma^l$  for  $\gamma \in \Gamma$ . Finally, set  $\Lambda^c\Gamma = \mathbb{Z}_l^c \otimes_{\mathbb{Z}_l} \Lambda\Gamma$ , with  $\mathbb{Z}_l^c$  the ring of integers in  $\mathbb{Q}_l^c$ .

With this notation we specify the subgroup

$$\text{HOM}(R_l(G), (\Lambda^c\Gamma_k)^\times) \quad \text{of} \quad \text{Hom}(R_l(G), (\Lambda^c\Gamma_k)^\times)$$

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<sup>6</sup>so, in particular,  $S$  contains the primes dividing  $l$

to consist of all homomorphisms  $f : R_l(G) \rightarrow (\Lambda^c \Gamma_k)^\times$  satisfying

1. equivariance with respect to the natural action of  $G_{\mathbb{Q}_l^c/\mathbb{Q}_l}$  on  $R_l(G)$  and  $\Lambda^c \Gamma_k$ ,
2.  $f(\chi \otimes \rho) = \rho^\sharp(f(\chi))$  ( $\forall \chi$  and all  $\rho$  of type W),
3.  $\frac{f(\chi)^l}{\Psi f(\psi_l \chi)} \equiv 1 \pmod{l \Lambda^c \Gamma_k}$  ( $\forall \chi$ ).

FACT 1. *The determinant  $\text{Det} : K_1(\mathcal{Q}G) \rightarrow \text{Hom}(R_l(G), (\mathcal{Q}^c \Gamma_k)^\times)$  induces*

$$\text{Det} : K_1(\Lambda G) \rightarrow \text{HOM}(R_l(G), (\Lambda^c \Gamma_k)^\times).$$

Above,  $\Lambda G, \Lambda^c \Gamma_k$  may be replaced by  $\Lambda_\bullet G, \Lambda_\bullet^c \Gamma_k$  or by  $\Lambda_\wedge G, \Lambda_\wedge^c \Gamma_k$ , where  $\Lambda_\wedge -$  is the  $l$ -adic completion of  $\Lambda_\bullet -$ .

NB The appearance of  $\Lambda_\bullet -$  has already been observed in the abelian case. Since we will have to work with logarithms, we need to complete in order to guarantee the convergence of the logarithm series.

For the actual definition of  $\text{Det} : K_1(\mathcal{Q}G) \rightarrow \text{Hom}(R_l(G), (\mathcal{Q}^c \Gamma_k)^\times)$  see [RW2, p.558]; for FACT 1 compare [RW3, Lemma 2, Propositions 4 and 11]. Explicitly, we will use the formula

$$(\text{Det } g)(\chi) = \chi(g) \overline{g} \quad \text{for } g \in G \quad \text{with image } \overline{g} \in \Gamma_k$$

whenever  $\chi$  is irreducible of degree 1 [RW3, Proposition 5].

Introducing  $T(\mathcal{Q}G) = \mathcal{Q}G/[\mathcal{Q}G, \mathcal{Q}G]$ , with  $[\mathcal{Q}G, \mathcal{Q}G]$  the additive subgroup generated by all  $ab - ba$ ,  $a, b \in \mathcal{Q}G$ , we obtain the trace isomorphism <sup>7</sup>

$$\text{Tr} : T(\mathcal{Q}G) \rightarrow \text{Hom}^*(R_l(G), \mathcal{Q}^c \Gamma_k)$$

given by  $\text{Tr}(\tau g)(\chi) = \chi(g) \overline{g}$ , where  $\tau g$  denotes the image of  $g \in G$  in  $T(\mathcal{Q}G)$  (compare [RW3, Lemma 6, Proposition 3]). We remark that the natural map  $T(\Lambda G) \rightarrow T(\mathcal{Q}G)$  is injective.

From now on  $G$  is always a pro- $l$  group. Define

$$\mathbf{L} : \text{HOM}(R_l(G), (\Lambda^c \Gamma_k)^\times) \rightarrow \text{Hom}^*(R_l(G), \mathcal{Q}^c \Gamma_k) \quad \text{by } f \mapsto [\chi \mapsto \frac{1}{l} \log \frac{f(\chi)^l}{\Psi f(\psi_l \chi)}].$$

FACT 2. *The homomorphism  $\mathbf{L}$  induces a unique homomorphism  $\mathbb{L} : K_1(\Lambda G) \rightarrow T(\mathcal{Q}G)$  making the square*

$$\begin{array}{ccc} K_1(\Lambda G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}G) \\ \text{Det} \downarrow & & \text{Tr}, \simeq \downarrow \\ \text{HOM}(R_l(G), (\Lambda^c \Gamma_k)^\times) & \xrightarrow{\mathbf{L}} & \text{Hom}^*(R_l(G), \mathcal{Q}^c \Gamma_k) \end{array}$$

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<sup>7</sup>The superscript  $*$  means HOM without condition 3. In [RW3] HOM,  $\Lambda$  and functions  $f$  often carry superscripts such as  $*, N, \mathfrak{D}, \text{Fr}$ : since we are going to restrict  $G$  to be a pro- $l$  group, these exponents should just be ignored.

commute. This  $\mathbb{L}$  takes  $K_1(\Lambda G)$  into  $T(\Lambda G)$ . The same holds with  $\Lambda G, \Lambda^\circ \Gamma_k, \mathcal{Q}G, \mathcal{Q}^\circ \Gamma_k$  replaced by  $\Lambda_\wedge G, \Lambda_\wedge^\circ \Gamma_k, \mathcal{Q}_\wedge G, \mathcal{Q}_\wedge^\circ \Gamma_k$ , respectively <sup>8</sup>.

For this, see [RW3, Theorem 8, Proposition 11].

We finally turn to the Iwasawa  $L$ -function  $L_{K/k}$ . For this, we first fix a finite set  $S$  of places of  $k$  which contains all archimedean places and all places of  $k$  above  $l$ . For  $\chi \in R_l(G)$ , let  $L_{l,S}(s, \chi)$  be the  $l$ -adic Artin  $L$ -function with respect to  $S$ , as defined in [Gr]. By [C-N] or [DR] there is an element  $G_{\chi,S}(T) \in \mathbb{Q}_l^\times \otimes_{\mathbb{Q}_l} \text{Quot} \mathbb{Z}_l[[T]]$  such that  $L_{l,S}(1-s, \chi) = \frac{G_{\chi,S}(u^s-1)}{H_\chi(u^s-1)}$ , with the 1-unit  $u \in \mathbb{Z}_l$  that has already appeared in the introduction and with  $H_\chi(T) = \chi(\gamma_k)(1+T) - 1$  or  $= 1$  according as  $H \leq \ker \chi$  or not.

With this notation  $L_{K/k}$  is defined by  $L_{K/k}(\chi) = \frac{G_{\chi,S}(\gamma_k-1)}{H_\chi(\gamma_k-1)}$ . Note that although  $L_{K/k}$  depends on  $S$  we suppress this dependence in our notation; note also that  $L_{K/k}$  is independent of a special choice of  $\gamma_k$  [RW2, Proposition 11].

**FACT 3.** *If  $S$  is sufficiently large and if Iwasawa's  $\mu$ -invariant of  $K/k$  vanishes,  $L_{K/k} \in \text{HOM}(R_l(G), (\Lambda_\bullet^\circ \Gamma_k)^\times)$ . Moreover,  $L_{K/k} \in \text{Det } K_1(\Lambda_\bullet G)$  if, and only if,  $L_{K/k} \in \text{Det } K_1(\Lambda_\wedge G)$ .*

For this see [RW3, Corollary of Theorem 9] and compare Theorems A in [RW3] and [RW4]. We remark that the assumption  $\mu = 0$  is independent of the size of  $S$  [NSW, (11.3.6)], because no prime of  $k$  splits completely in the cyclotomic  $\mathbb{Z}_l$ -extension  $k_\infty$  of  $k$ .

From now on we assume  $\mu = 0$  for the extension  $K/k$ .

## 2. THE LOGARITHMIC PSEUDOMEASURE AND INTEGRALITY

We start out from the diagram shown in FACT 2, but now read in the completed situation :

$$\begin{array}{ccc} K_1(\Lambda_\wedge G) & \xrightarrow{\mathbb{L}} & T(\mathcal{Q}_\wedge G) \\ \text{Det} \downarrow & & \text{Tr}, \simeq \downarrow \\ \text{HOM}(R_l(G), (\Lambda_\wedge^\circ \Gamma_k)^\times) & \xrightarrow{\mathbf{L}} & \text{Hom}^*(R_l(G), \mathcal{Q}_\wedge^\circ \Gamma_k). \end{array}$$

**DEFINITION.**  $t_{K/k} \in T(\mathcal{Q}_\wedge G)$  is the unique element satisfying  $\text{Tr}(t_{K/k}) = \mathbf{L}(L_{K/k})$ ; we call it the logarithmic pseudomeasure of  $K/k$ .

Recall that we are assuming  $\mu = 0$  for the pro- $l$  extension  $K/k$  and note that then  $t_{K/k} = \mathbb{L}(\lambda_{K/k})$  if  $G$  is abelian (compare [RW3, Corollary to Theorem 9]).

**LEMMA 2.1.** *If  $N$  is a finite normal subgroup of  $G$  with fixed field  $\tilde{K}$ , then  $\text{defl}_{\tilde{K}}^{\tilde{G}} t_{K/k} = t_{\tilde{K}/k}$ , where  $\tilde{G} = G_{\tilde{K}/k} (\simeq G/N)$ .*

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<sup>8</sup>Contrary to [RW3], we prefer to write  $\Lambda_\wedge -$  rather than  $(\Lambda -)_\wedge$  and the same with the total ring of fractions  $\mathcal{Q}_\wedge -$  of  $\Lambda_\wedge -$ .

To see this, we recall that on the  $K_1$ -level deflation is induced by  $\Lambda_\wedge \tilde{G} \otimes_{\Lambda_\wedge G} -$ , on the  $T$ -level by  $G \rightarrow \tilde{G}$ , and on the Hom-level by  $f \mapsto [\tilde{\chi} \mapsto f(\text{infl}_{\tilde{G}}^G \tilde{\chi})]$  for  $\tilde{\chi} \in R_l(\tilde{G})$ . By [RW2, Lemma 9],  $\text{Det}$  and  $\text{defl}_{\tilde{G}}^{\tilde{G}}$  commute and it is shown below that also  $\text{Tr}$  and  $\text{defl}_{\tilde{G}}^{\tilde{G}}$  commute. Finally,  $\mathbf{L}$  and  $\text{defl}_{\tilde{G}}^{\tilde{G}}$  commute because  $\psi_l$  and  $\text{infl}_{\tilde{G}}^G$  do. Hence the lemma follows from  $\text{defl}_{\tilde{G}}^{\tilde{G}}(L_{K/k}) = L_{\tilde{K}/k}$  (see [RW2, p.563]).

In order to check commutativity of the diagram

$$\begin{array}{ccc} T(\mathcal{Q}_\wedge G) & \xrightarrow{\text{defl}} & T(\mathcal{Q}_\wedge \tilde{G}) \\ \text{Tr} \downarrow & & \tilde{\text{Tr}} \downarrow \\ \text{Hom}^*(R_l(G), \mathcal{Q}_\wedge^c \Gamma_k) & \xrightarrow{\text{defl}} & \text{Hom}^*(R_l(\tilde{G}), \mathcal{Q}_\wedge^c \Gamma_k), \end{array}$$

we choose a central open subgroup  $\Gamma$  in  $G$  and representatives  $g_i \in G$ ,  $1 \leq i \leq s$ , of the conjugacy classes of the finite group  $G/\Gamma$ . Writing  $\sim$  for the map  $G \rightarrow \tilde{G}$ , take  $x \in T(\mathcal{Q}_\wedge G)$  and a character  $\tilde{\chi}$  of  $\tilde{G}/\tilde{\Gamma} \simeq G/\Gamma N$ . Writing  $x = \sum_{i=1}^s x_i \tau(g_i)$  with  $x_i \in \mathcal{Q}_\wedge \Gamma$  (see [RW3, Lemma 5]) and  $\chi = (\text{infl})(\tilde{\chi})$ , we compute  $(\tilde{\text{Tr}} \circ \text{defl})(x)(\tilde{\chi}) = \tilde{\text{Tr}}(\sum_i \tilde{x}_i \tau(\tilde{g}_i))(\tilde{\chi}) = \sum_i \tilde{x}_i \tilde{g}_i \tilde{\chi}(\tilde{g}_i) = \sum_i \tilde{x}_i \tilde{g}_i \chi(g_i)$  and  $(\text{defl} \circ \text{Tr})(x)(\tilde{\chi}) = \text{defl}(\text{Tr } x)(\tilde{\chi}) = (\text{Tr } x)(\text{infl } \tilde{\chi}) = (\text{Tr } x)(\chi) = \sum_i \tilde{x}_i \tilde{g}_i \chi(g_i)$ . Thus  $(\tilde{\text{Tr}} \circ \text{defl})(x)$  and  $(\text{defl} \circ \text{Tr})(x)$  agree on irreducible characters  $\tilde{\chi}$  of  $\tilde{G}/\tilde{\Gamma}$ . Since every irreducible character of  $\tilde{G}$  is a twist  $\tilde{\chi} \otimes \tilde{\rho}$  with  $\tilde{\rho}$  of type W (compare [RW4, p.164, proof of 2. of Lemma 4]), checking compatibility with W-twisting then gives  $(\tilde{\text{Tr}} \circ \text{defl})(x) = (\text{defl} \circ \text{Tr})(x)$ . The lemma is established.

**PROPOSITION 2.2.** *If  $t_{K/k} \in T(\Lambda_\wedge G)$ , then  $t_{K/k}$  is in  $\mathbb{L}(K_1(\Lambda_\wedge G))$ . Moreover, there exists a  $y \in (\Lambda_\wedge G)^\times$  so that  $\mathbb{L}(y) = t_{K/k}$  and  $\text{defl}_G^{G^{\text{ab}}}(y) = \lambda_{K_{\text{ab}}/k}$ , where  $K_{\text{ab}}$  is the fixed field of  $[G, G] (\leq H)$  and  $G^{\text{ab}} = G/[G, G]$ .*

The proof depends on the commutative diagram

$$\begin{array}{ccccc} 1 + \mathfrak{a}_\wedge & \hookrightarrow & (\Lambda_\wedge G)^\times & \xrightarrow{\text{defl}} & (\Lambda_\wedge G^{\text{ab}})^\times \\ \downarrow & & \mathbb{L} \downarrow & & \mathbb{L}^{\text{ab}} \downarrow \\ \tau(\mathfrak{a}_\wedge) & \hookrightarrow & T(\Lambda_\wedge G) & \xrightarrow{\text{defl}} & \Lambda_\wedge G^{\text{ab}} \end{array}.$$

To understand the diagram, we recall that the canonical map  $(\Lambda_\wedge G)^\times \rightarrow K_1(\Lambda_\wedge G)$  is surjective and that  $(\Lambda_\wedge G^{\text{ab}})^\times = K_1(\Lambda_\wedge G^{\text{ab}})$  (see [CR, 40.31 and 45.12]). Whence we may regard  $\mathbb{L}$  as defined on  $(\Lambda_\wedge G)^\times$ .

In the diagram,  $1 + \mathfrak{a}_\wedge$  is the kernel of the upper  $\text{defl}$ . This map is surjective since  $\mathfrak{a}_\wedge \subset \text{rad}(\Lambda_\wedge G)$ . The surjectivity of the lower  $\text{defl}$  is obvious. Finally, by [RW3, 2b. of Proposition 11] the left vertical map is surjective. The snake lemma shows that  $\text{coker}(\mathbb{L})$  and  $\text{coker}(\mathbb{L}^{\text{ab}})$  are isomorphic and that  $\ker(\mathbb{L})$  maps onto  $\ker(\mathbb{L}^{\text{ab}})$ . Since  $t_{K_{\text{ab}}/k} = \mathbb{L}^{\text{ab}}(\lambda_{K_{\text{ab}}/k})$  and  $\text{defl}(t_{K/k}) = t_{K_{\text{ab}}/k}$ , the first assertion follows.

So  $\mathbb{L}(y) = t_{K/k}$  for some  $y \in (\Lambda_\wedge G)^\times$ . Then  $\text{defl}(y)^{-1} \lambda_{K_{\text{ab}}/k} \in \ker(\mathbb{L}^{\text{ab}})$ , by the diagram, hence it equals  $\text{defl}(z)$  with  $z \in \ker(\mathbb{L})$ . Thus  $\text{defl}(yz) = \lambda_{K_{\text{ab}}/k}$  and  $\mathbb{L}(yz) = t_{K/k}$ .

The proof of Proposition 2.2 is complete.

To continue the discussion of the integrality of  $t_{K/k}$ , we again pick a central open subgroup  $\Gamma$  in  $G$  and representatives  $g_1, \dots, g_s$  in  $G$  of the conjugacy classes of the group  $G/\Gamma$ ; so the images  $\tau(g_i)$  of  $g_i$  in  $T(\Lambda_\wedge G)$  constitute a  $\Lambda_\wedge \Gamma$ -basis of  $T(\Lambda_\wedge G)$  (see [RW3, Lemma 5 $_\wedge$ ]). We will employ the following notation :

*given  $t \in T(\mathcal{Q}_\wedge G)$ , there is a unique function  $(t|\Gamma) : G \rightarrow \mathcal{Q}_\wedge \Gamma$  such that*

$$(t|\Gamma)(g) = (t|\Gamma)(g') \text{ if } g \text{ and } g' \text{ are conjugate in } G,$$

$$(t|\Gamma)(\gamma g) = \gamma(t|\Gamma)(g) \text{ for } \gamma \in \Gamma,$$

$$t = \sum_{i=1}^r (t|\Gamma)(g_i^{-1}) \tau(g_i) \text{ with the } g_i \text{ as above.}$$

This is easily checked. We call  $(t|\Gamma)(g^{-1})$  the *coefficient* of  $t$  at  $g$ . In particular,  $[t_{K/k} \in T(\Lambda_\wedge G) \iff (t_{K/k}|\Gamma)(g^{-1}) \in \Lambda_\wedge \Gamma \text{ for all } g \in G]$ .

We now compute  $(t_{K/k}|\Gamma)(g^{-1})$ . From  $\text{Tr}(t_{K/k}) = \mathbf{L}(L_{K/k})$  we obtain

$$\sum_{i=1}^r \overline{(t_{K/k}|\Gamma)(g_i^{-1})} \bar{g}_i \chi_j(g_i) = \frac{1}{l} \log \frac{L_{K/k}(\chi_j)^l}{\Psi L_{K/k}(\psi_l \chi_j)}$$

for the irreducible characters  $\chi_j$ ,  $1 \leq j \leq s$ , of  $G/\Gamma$ . This equation enables us to perform a Fourier inversion in order to isolate the  $\overline{(t_{K/k}|\Gamma)(g_i^{-1})}$ . To that end, let  $h_i$  denote the order of the conjugacy class of  $(g_i \bmod \Gamma)$  and denote by  $M, M_1$  the  $s \times s$ -matrices  $(\chi_i(g_j))_{i,j}, (h_i \chi_j(g_i^{-1}))_{i,j}$ , respectively. The orthogonality relations yield  $M \cdot M_1 = [G : \Gamma] \cdot \mathbf{1}$  whence  $M_1 \cdot M = [G : \Gamma] \cdot \mathbf{1}$  as well, and we arrive first at

$$M \cdot \begin{pmatrix} \vdots \\ \overline{(t_{K/k}|\Gamma)(g_i^{-1})} \bar{g}_i \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \frac{1}{l} \log \frac{L_{K/k}(\chi_j)^l}{\Psi L_{K/k}(\psi_l \chi_j)} \\ \vdots \end{pmatrix}$$

and then at

$$[G : \Gamma] \overline{(t_{K/k}|\Gamma)(g_i^{-1})} \bar{g}_i = \sum_{j=1}^s h_i \chi_j(g_i^{-1}) \frac{1}{l} \log \frac{L_{K/k}(\chi_j)^l}{\Psi L_{K/k}(\psi_l \chi_j)}, \quad (1 \leq i \leq s).$$

So we have

$$\text{PROPOSITION 2.3. } \overline{(t_{K/k}|\Gamma)(g_i^{-1})} = \frac{1}{[G:\Gamma]} \sum_{j=1}^s h_i \chi_j(g_i^{-1}) \frac{1}{l} \log \frac{L_{K/k}(\chi_j)^l}{\Psi L_{K/k}(\psi_l \chi_j)} \bar{g}_i^{-1}$$

Observe that the right hand side is in  $\mathcal{Q}_\wedge \Gamma_k$  (by Galois invariance); it is in  $\Lambda_\wedge \Gamma_k$  for all  $g_i$  precisely when  $t_{K/k} \in T(\Lambda_\wedge G)$ . Recall also that, up to W-twists, the  $\chi_j$  are a full set of representatives of all irreducible characters of  $G$ .

**PROPOSITION 2.4.** *1. If  $t_{K/k} \in T(\Lambda_\wedge G)$ , then there exists a torsion element*

$$w \in \text{HOM}(R_l(G), (\Lambda_\wedge^\Gamma \Gamma_k)^\times) \text{ so that } \text{def}_G^{\text{G}^{\text{ab}}}(w) = 1 \text{ and } w \cdot L_{K/k} \in \text{Det } K_1(\Lambda_\wedge G).$$

2. If  $w \in \text{HOM}(R_l(G), (\Lambda_\wedge^c \Gamma_k)^\times)$  is torsion and  $w \cdot L_{K/k} \in \text{Det } K_1(\Lambda_\wedge G)$ , then  $t_{K/k} \in T(\Lambda_\wedge G)$ .
3. There is at most one torsion element  $w \in \text{HOM}(R_l(G), (\Lambda_\wedge^c \Gamma_k)^\times)$  so that  $\text{defl}_G^{G^{\text{ab}}}(w) = 1$  and  $w \cdot L_{K/k} \in \text{Det } K_1(\Lambda_\wedge G)$ .

For 1., take a  $y$  with  $\mathbb{L}(y) = t_{K/k}$  from Proposition 2.2 and define  $w$  by  $w \cdot L_{K/k} = \text{Det}(y)$ . Applying  $\text{defl} = \text{defl}_G^{G^{\text{ab}}}$ , we get

$$\text{defl}(w) L_{K_{\text{ab}}/k} = \text{Det}(\text{defl } y) = \text{Det}(\lambda_{K_{\text{ab}}/k}) = L_{K_{\text{ab}}/k},$$

hence  $\text{defl}(w) = 1$ . Next apply **L** of FACT 2 and get

$$\mathbf{L}(w) + \text{Tr}(t_{K/k}) = \mathbf{L}(\text{Det } y) = \text{Tr}(\mathbb{L} y) = \text{Tr}(t_{K/k}),$$

hence  $\mathbf{L}(w) = 0$ . By the argument in [RW3, §6, just before the Corollary of Theorem B $_\wedge$ ], showing that  $\text{Det } z$  is torsion, it follows that  $w$  is torsion.

2. follows from the  $\wedge$ -version of FACT 2 and  $\mathbf{L}(w) = 0$ .

For 3., suppose  $w_1, w_2$  are both as above, hence  $w_i L_{K/k} = \text{Det}(y_i)$  with  $y_i \in (\Lambda_\wedge G)^\times$  and  $\text{defl}(w_i) = 1$ . Then  $w_1^{-1} w_2 = \text{Det}(y_1^{-1} y_2)$  and  $\text{defl}(w_1^{-1} w_2) = 1$ . Therefore,  $\text{Det}(\text{defl}(y_1^{-1} y_2)) = 1$ , since  $\text{Det}$  commutes with  $\text{defl}$ , from which  $\text{defl}(y_1^{-1} y_2) = 1$  follows because  $\text{Det}$  is an isomorphism by  $SK_1(\Lambda_\wedge G^{\text{ab}}) = 1$ . So  $y_1^{-1} y_2 \in 1 + \mathfrak{a}_\wedge$ , by the diagram in the proof of Proposition 2.2, and thus  $w_1^{-1} w_2 \in \text{Det}(1 + \mathfrak{a}_\wedge)$  is torsion and therefore 1 by [RW3, Lemma 12].

The proof is finished.

REMARK. It may be worth mentioning that in Fröhlich's work on Galois module structure logarithmic and torsion congruences appear as well: see [Fr, IV, §§4,5,6]. Here, the logarithmic congruences are taken care of by the Davenport-Hasse formula [Fr, p.179].

### 3. THE TORSION CONGRUENCE

In this section  $S$  is sufficiently large. We continue to assume  $\mu = 0$  for the pro- $l$  extension  $K/k$  and, under the hypothesis  $t_{K/k} \in T(\Lambda_\wedge G)$ , exhibit the additional congruences which are equivalent to  $L_{K/k} \in \text{Det } K_1(\Lambda_\wedge G)$  or, by FACT 3, to  $L_{K/k} \in \text{Det } K_1(\Lambda_\bullet G)$ .

Assume first that there is an abelian subgroup  $G'$  of index  $l$  in the non-abelian pro- $l$  group  $G$ . Denote by  $k'$  the fixed field of  $G'$  and by  $\text{ver}$  the transfer map  $G^{\text{ab}} \rightarrow G'$ . Setting  $A = G/G'$ , consider the conjugation action of  $A$  on  $G'$  and  $\Lambda_\wedge G'$ , and let  $\mathcal{T}'$  be the image of the  $A$ -trace map on  $\Lambda_\wedge G'$ . Thus  $\mathcal{T}'$  is an ideal in the ring  $(\Lambda_\wedge G')^A$  of  $A$ -fixed points of  $\Lambda_\wedge G'$ .

LEMMA 3.1.  $\lambda_{K/k'}$  is  $A$ -invariant.

This is because  $\text{Det} : K_1(\Lambda_\bullet G') \rightarrow \text{HOM}(R_l(G'), \Lambda_\bullet^c(\Gamma_{k'})^\times)$  is an  $A$ -equivariant monomorphism (see [RW4, 2nd paragraph on p.159]) and  $\text{Det } \lambda_{K/k'} = L_{K/k'}$ ,  $L_{K/k'}^a(\chi') = L_{K/k'}(\chi'^{a^{-1}}) = L_{K/k}(\text{ind}_{G'}^G(\chi'^{a^{-1}})) = L_{K/k}(\text{ind}_{G'}^G(\chi')) = L_{K/k'}(\chi')$  for all  $a \in A$  and all  $\chi' \in R_l(G')$ .



PROPOSITION 3.2. *In the above situation and with  $w$  as in Proposition 2.4, the following are equivalent*

1.  $w = 1$ , i.e.,  $L_{K/k} \in \text{Det } K_1(\Lambda_\wedge G)$ ,
2.  $\text{ver}(\lambda_{K_{\text{ab}}/k}) \equiv \lambda_{K/k'} \pmod{\mathcal{T}'}$ ,
3.  $\frac{1}{[G':\Gamma]} \sum_{\chi'} \frac{L_{K/k'}(\chi')}{\Psi_{L_{K/k}(\chi') \text{ over}}} \equiv 1 \pmod{l \cdot \Lambda_\wedge^c \Gamma_k}$ , with the sum ranging over the  $\mathbb{Q}_l^c$ -irreducible characters  $\chi'$  of  $G'/\Gamma$  where  $\Gamma$  is a central open subgroup of  $G$ .

PROOF. Note first that  $w = 1$  if, and only if,  $\text{res}_G^{G'} w = 1$ : If  $\chi$  is an irreducible character of  $G$ , then either  $\chi = \text{infl}_{G_{\text{ab}}}^G \alpha$  is inflated from an abelian character  $\alpha$  of  $G^{\text{ab}}$  or  $\chi = \text{ind}_{G'}^G(\chi')$  is induced from an abelian character  $\chi'$  of  $G'$ . Therefore, either  $w(\chi) = w(\text{infl } \alpha) = (\text{defl } w)(\alpha) = 1$  or  $w(\chi) = w(\text{ind } \chi') = (\text{res } w)(\chi') = 1$ .

Write  $w \cdot L_{K/k} = \text{Det } y$  with  $y \in (\Lambda_\wedge G)^\times$  as in the proof of Proposition 2.4, so that  $\text{defl}_G^{G^{\text{ab}}} y = \lambda_{K_{\text{ab}}/k}$ . Since  $\text{res}_G^{G'}$  commutes with  $\text{Det}$  and  $\text{res}_G^{G'} L_{K/k} = L_{K/k'}$  (see [RW2, Lemma 9 and p.563]), we have

$$\text{res}_G^{G'} w = \text{Det} \left( \frac{\text{res}_G^{G'} y}{\lambda_{K/k'}} \right).$$

Since  $\text{Det}$  is injective on  $(\Lambda_\wedge G')^\times$ , it follows that  $\frac{\text{res}_G^{G'} y}{\lambda_{K/k'}}$  is a torsion element.

By means of the commutative square

$$\begin{array}{ccc} (\Lambda_\wedge G)^\times & \twoheadrightarrow & K_1(\Lambda_\wedge G) \\ N \downarrow & & \text{res}_G^{G'} \downarrow \\ (\Lambda_\wedge G')^\times & = & K_1(\Lambda_\wedge G') \end{array}$$

we write  $\text{res}_G^{G'} y = N(y)$ , the determinant of right multiplication of  $y$  on the left  $\Lambda_\wedge G'$ -module  $\Lambda_\wedge G$  (compare [RW3, proof of Lemma 12]).

Denoting the composite map  $G \xrightarrow{\text{defl}} G^{\text{ab}} \xrightarrow{\text{ver}} G'$  also by  $\text{ver}$ , we obtain from  $\text{defl } y = \lambda_{K_{\text{ab}}/k}$  the trivial equation

$$\frac{\text{res}_G^{G'} y}{\lambda_{K/k'}} = \frac{N(y)}{\text{ver}(y)} \cdot \frac{\text{ver}(\lambda_{K_{\text{ab}}/k})}{\lambda_{K/k'}}.$$

By a congruence due to C.T.C. Wall,  $N(y) \equiv \text{ver}(y) \pmod{\mathcal{T}'}$  with  $\mathcal{T}'$  the  $A$ -trace ideal of the  $A$ -action on  $\Lambda_\wedge G'$  (see [RW3, proof of Lemma 12]). Since both,  $N(y)$  and  $\text{ver}(y)$ , are units in  $\Lambda_\wedge G'$  fixed by  $A$ , it follows that

$$\frac{\text{res}_G^{G'} y}{\lambda_{K/k'}} \equiv \frac{\text{ver}(\lambda_{K_{\text{ab}}/k})}{\lambda_{K/k'}} \pmod{\mathcal{T}'}$$

The equivalence of 1. and 2. now follows from the last two paragraphs if we can show for torsion units  $e$  of  $\Lambda_\wedge G'$  that

$$e \equiv 1 \pmod{\mathcal{T}'} \iff e = 1.$$

But this has already been shown in the fifth paragraph of the proof of [RW3, Lemma 12]: the argument

$$N(x) \text{ torsion} \ \& \ N(x) \equiv 1 \pmod{\mathcal{T}'} \implies N(x) = 1$$

works with  $N(x)$  replaced by  $e$ .

We next turn to the equivalence of 2. and 3. and first prove, for irreducible characters  $\chi'$  of  $G'/\Gamma$ ,

$$\text{Det}\left(\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})}\right)(\chi') = \frac{L_{K/k'}(\chi')}{\Psi(L_{K/k}(\chi' \circ \text{ver}))}.$$

With  $\lambda_{K_{\text{ab}}/k} = \text{defl } y$  as before, hence  $\text{ver}(\lambda_{K_{\text{ab}}/k}) = \text{ver}(y)$ , write  $y = \sum_x y_x x$  with  $y_x \in \Lambda_\wedge \Gamma$  and  $\{x\}$  a set of representatives of the elements of  $G/\Gamma$  in  $G$ . Since  $\text{ver}$  induces  $\Psi$  on  $\Gamma$ , we obtain  $\text{ver}(y) = \sum_x \Psi(y_x) \text{ver}(x)$ , so

$$(\text{Det}(\text{ver } y))(\chi') = \sum_x \overline{\Psi(y_x)} \bar{x}^l \chi'(\text{ver } x)$$

because  $\overline{\text{ver}(x)} = \bar{x}^l$  (compare the first displayed formula after FACT 1).

On the other hand,  $(\text{Det } y)(\chi' \circ \text{ver}) = \sum_x \bar{y}_x \bar{x}(\chi' \circ \text{ver})(x)$ , whence  $\Psi(\text{Det } y)(\chi' \circ \text{ver}) = \sum_x \Psi(\bar{y}_x) \bar{x}^l(\chi' \circ \text{ver})(x)$ . Thus  $(\text{Det}(\text{ver } y))(\chi') = \Psi((\text{Det } y)(\chi' \circ \text{ver}))$  and we conclude

$$(\text{Det} \frac{\lambda_{K/k'}}{\text{ver}(y)})(\chi') = \frac{L_{K/k'}(\chi')}{\Psi((\text{Det } y)(\chi' \circ \text{ver}))} = \frac{L_{K/k'}(\chi')}{\Psi((w \cdot L_{K/k})(\chi' \circ \text{ver}))} = \frac{L_{K/k'}(\chi')}{\Psi((L_{K/k})(\chi' \circ \text{ver}))}$$

as  $\text{defl}_G^{G^{\text{ab}}} w = 1$ .

Now write  $\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})} = \sum_{j=1}^s a_j g_j$  with  $a_j \in \Lambda_\wedge \Gamma$  and  $1 = g_1, \dots, g_s$  a set of representatives of the elements of  $G'/\Gamma$  in  $G'$ , of which the first  $r$  are precisely those in the centre  $Z(G)$  of  $G$ . Then

$$\text{Det}\left(\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})}\right)(\chi') = \sum_j \bar{a}_j \bar{g}_j \chi'(g_j),$$

and by Fourier inversion, in view of the above paragraph,

$$\bar{a}_j \bar{g}_j = \frac{1}{[G' : \Gamma]} \sum_{\chi'} \chi'(g_j^{-1}) \frac{L_{K/k'}(\chi')}{\Psi(L_{K/k}(\chi' \circ \text{ver}))}.$$

Since, by Lemma 3.1,  $\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})}$  is  $A$ -invariant, the coefficients  $a_j$  are constant on orbits of the  $A$ -action on  $G'$ , hence  $\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})} \equiv \sum_{j=1}^r a_j g_j \pmod{\mathcal{T}'}$ . Note that by the fourth paragraph<sup>9</sup> of the proof of [RW3, Lemma 12]  $g_1 + \mathcal{T}', \dots, g_r + \mathcal{T}'$  is a  $\Lambda_\wedge \Gamma / l\Lambda_\wedge \Gamma$ -basis of  $(\Lambda_\wedge G')^A / \mathcal{T}'$ .

The fifth paragraph of that proof shows that  $\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})} \equiv \zeta z \pmod{\mathcal{T}'}$ , with a root  $\zeta$  of unity and a  $z \in Z(G)$  of finite order. If  $\zeta z \equiv \sum_{j=1}^r a_j g_j \pmod{\mathcal{T}'}$ , then there is a unique  $j_0$  so that  $\zeta z \equiv a_{j_0} g_{j_0}$  and all  $a_j g_j \equiv 0$  for  $j \neq j_0$  (this follows by writing  $z = \gamma g_{j_0}$  with  $\gamma \in \Gamma$ ). Hence, 2. is equivalent to  $a_1 \equiv 1 \pmod{l\Lambda_\wedge \Gamma}$  (since this implies  $j_0 = 1$ ). But the latter is equivalent to  $\bar{a}_1 \equiv 1 \pmod{l\Lambda_\wedge \bar{\Gamma}}$ , thus to  $\bar{a}_1 \equiv 1 \pmod{l\Lambda_\wedge \Gamma_k}$  as  $\Lambda_\wedge \bar{\Gamma} \cap l\Lambda_\wedge \Gamma_k = l\Lambda_\wedge \bar{\Gamma}$  where  $\Lambda_\wedge \Gamma_k$  is a free  $\Lambda_\wedge \bar{\Gamma}$ -module.

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<sup>9</sup>with care taken to choose the coset representatives  $\{b\}$  closed under conjugation by  $a$ ; this is possible since  $(b\Gamma)^a = b\Gamma$  implies  $b^a = b$ , as  $b^{a^{-1}} \in \Gamma \cap [G, G] = 1$

REMARK. The implication  $[2. \implies 3.]$  does not need any hypothesis as it depends on the Fourier inversion step in the above proof. Thus 3. can be restated as

$$\frac{1}{[G' : \Gamma]} \sum_{\chi'} \chi'(z^{-1}) \left( \frac{L_{K/k'}(\chi')}{\Psi_{L_{K/k}}(\chi' \circ \text{ver})} - 1 \right) \equiv 0 \pmod{l \cdot \Lambda_{\wedge}^c \Gamma_k} \quad (\forall z \in Z(G)),$$

since the coefficients of  $\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})} - 1$  at the central elements are divisible by  $l$ .

We finally free ourselves from the assumption that  $G$  has an abelian subgroup of index  $l$ .

THEOREM. Assume  $\mu = 0$  for  $K/k$  and  $t_{K/k} \in T(\Lambda_{\wedge} G)$ . Then  $L_{K/k} \in \text{Det } K_1(\Lambda_{\wedge} G)$  provided that  $L_{F/f} \in \text{Det } K_1(\Lambda_{\wedge} G_{F/f})$  for all intermediate Galois extensions  $F/f$  in  $K/k$  such that

1.  $G_{F/f}$  has an abelian subgroup of index  $l$ ,
2.  $[K : F]$  is finite and  $f$  is fixed by the centre  $Z(G)$  of  $G$ .

The proof is by induction on  $[G : Z(G)]$ .

By 1. of Proposition 2.4, there exists a torsion  $w \in \text{HOM}(R_l(G), (\Lambda_{\wedge}^c \Gamma_k)^{\times})$  so that  $\text{defl}_G^{G^{\text{ab}}} w = 1$  and  $w \cdot L_{K/k} \in \text{Det } K_1(\Lambda_{\wedge} G)$ . It suffices to show that  $w = 1$ .

Let  $G'$  be any subgroup of index  $l$  in  $G$  containing  $Z(G)$  and let  $k'$  be its fixed field. Note that all intermediate extensions of  $K/k'$  satisfying conditions 1. and 2. relative to  $K/k'$  also satisfy 1. and 2. relative to  $K/k$  because  $Z(G') \supset Z(G)$ . So the induction hypothesis implies  $L_{K/k'} \in \text{Det } K_1(\Lambda_{\wedge} G')$ .

Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}(K/k) & \xrightarrow{d} & \mathcal{H}(\tilde{K}/k) & \xrightarrow{q} & \mathcal{H}(K_{\text{ab}}/k) \\ \text{res}_G^{G'} \downarrow & & \text{res}_{\tilde{G}}^{G'^{\text{ab}}} \downarrow & & \\ \mathcal{H}(K/k') & \xrightarrow{d'} & \mathcal{H}(\tilde{K}/k') & & \end{array}$$

with  $\tilde{G} = G_{\tilde{K}/k} = G/[G', G']$  and  $\tilde{K}$  the fixed field of  $[G', G']$ . Here,  $\mathcal{H}(F/f)$  abbreviates  $\text{Hom}(R_l(G_{F/f}), (\Lambda_{\wedge}^c \Gamma_f)^{\times})$  and all horizontal maps are deflations<sup>10</sup>.

Now,

$$(dw)L_{\tilde{K}/k} = d(wL_{K/k}) \in \text{Det } K_1(\Lambda_{\wedge} \tilde{G}), \quad q(dw) = \text{defl}_{\tilde{G}}^{G^{\text{ab}}} w = 1$$

(by  $[G', G'] \subset [G, G]$ ). Since  $\tilde{G}$  has the abelian subgroup  $G'^{\text{ab}}$  of index  $l$ ,  $dw = 1$  follows by hypothesis 1. and 2.

Thus  $d'(\text{res}_G^{G'} w) = \text{res}_{\tilde{G}}^{G'^{\text{ab}}}(dw) = \text{res}_{\tilde{G}}^{G'^{\text{ab}}} 1 = 1$  and  $(\text{res}_G^{G'} w)L_{K/k'} \in \text{Det } K_1(\Lambda_{\wedge} G')$  (see [RW2, 2. of Proposition 12]). Hence, by the above,

$$\text{res}_G^{G'} w \in \text{Det } K_1(\Lambda_{\wedge} G') \subset \text{HOM}(R_l(G'), (\Lambda_{\wedge}^c \Gamma_{k'})^{\times}),$$

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<sup>10</sup>this is the analogue of the commutative diagram in [RW3, proof of Lemma 12]

by [RW3, 1. of Proposition 11] We can now invoke 3. of Proposition 2.4 to conclude that  $\text{res}_G^{G'} w = 1$ .

We now show  $w(\chi) = 1$  for every irreducible  $\chi$ . If  $\chi$  is abelian, then  $\text{defl}_G^{G^{\text{ab}}} w = 1$  takes care of this. If  $\chi$  is non-abelian, then it is induced from a character  $\chi'$  of a subgroup  $G'$  of index  $l$  in  $G$  which contains  $Z(G)$  (see [CR, 11.2]). Hence,  $w(\chi) = (\text{res}_G^{G'} w)(\chi') = 1$ , establishing the theorem.

REMARKS.

1. Combining the theorem with FACT 3 we even get  $L_{K/k} \in \text{Det } K_1(\Lambda_\bullet G)$ , hence  $K/k$  has a non-abelian pseudomeasure. Moreover, in all three statements in Proposition 3.2  $\wedge$  can be replaced by  $\bullet$ ; for 3. this is because  $\Lambda_\bullet \Gamma_k \cap l\Lambda_\wedge \Gamma_k = l\Lambda_\bullet \Gamma_k$ ; for 2. note that  $\mathcal{T}' \cap \Lambda_\bullet G'$  is the image of the  $A$ -trace map on  $\Lambda_\bullet G'$ .
2. The torsion congruence in Proposition 3.2 is somehow reminiscent of special value congruences which have been derived in [Ty1].

#### 4. MINIMAL NON-ABELIAN $G$

In this section  $G$  is a non-abelian pro- $l$  group which has an abelian subgroup  $G'$  of index  $l$ . Further,  $A$  is short for the cyclic group  $G/G'$  of order  $l$  and  $a \bmod G'$  is a generator. Finally, we fix an irreducible character  $\omega \in R_l(G)$  which is trivial on  $G'$  but not on  $a$  and a central open subgroup  $\Gamma$  of  $G$  (so  $\Gamma \leq Z(G) \leq G' \leq G$ ).

We continue to assume  $\mu = 0$  for  $K/k$  and  $S$  to be sufficiently large.

LEMMA 4.1. 1. If  $\chi'$  is an irreducible character of  $G'$ , then

$$\text{ind}_G^{G'}(\psi_l \chi') - \psi_l(\text{ind}_G^{G'} \chi') = \sum_{i=0}^{l-1} (\chi' \circ \text{ver}) \omega^i - l(\chi' \circ \text{ver})$$

with  $\text{ver}$  the transfer  $G \rightarrow G^{\text{ab}} \rightarrow G'$ ,

$$2. \frac{[G:Z(G)]}{l \cdot |[G,G]|} \in \mathbb{Z}.$$

The first assertion results on evaluating both sides on an element  $g \in G$  (recall  $\text{ver}(g) = \prod_{i=0}^{l-1} g^{a^i}$  or  $= g^l$  according as  $g \in G'$  or not, and  $\text{im}(\text{ver}) \subset Z(G)$ ).

For the second assertion we use induction on  $[G, G]$ . By  $[G, G] = [G, G']$  we can pick a central commutator  $z = [a, g']$  of order  $l$ ; we set  $\tilde{G} = G/\langle z \rangle$ . If  $[G, G] = l$ , then  $\langle z \rangle = [G, G]$  and the fraction in question is  $\frac{[G:Z(G)]}{l^2}$ , which is integral as  $G$  is non-abelian. If however  $[G, G] > l$ , then  $\tilde{G}$  is non-abelian, but  $[\tilde{G}, \tilde{G}] = [\widetilde{[G, G]}]$  and  $Z(\tilde{G}) \not\supseteq \widetilde{Z(G)}$ ; the latter by  $\tilde{g}' \in Z(\tilde{G}) \setminus \widetilde{Z(G)}$ .

Summing up and setting  $l^\nu = [Z(\tilde{G}) : \widetilde{Z(G)}]$ , we obtain  $\mathbb{Z} \ni \frac{[\tilde{G}:Z(\tilde{G})]}{l \cdot |[G,G]|} = \frac{[G:Z(G)]}{|[G,G]| \cdot l^\nu}$ , which finishes the proof of 2. and thus of the lemma.

We next turn to the computation of the coefficient  $(t_{K/k}|\Gamma)(g^{-1})$  at  $g \in G$  of the logarithmic pseudomeasure  $t_{K/k}$ . To ease notation, set  $m_\chi = \mathbf{L}(L_{K/k})(\chi) = \frac{1}{l} \log \frac{L_{K/k}(\chi)^l}{\Psi_{L_{K/k}}(\psi_l \chi)}$  whenever  $\chi$  is a character of  $G$  which is trivial on  $\Gamma$ . Obviously,  $m_{\chi_1 + \chi_2} = m_{\chi_1} + m_{\chi_2}$  and  $m_{\text{infl}_{\tilde{G}}^G \tilde{\chi}} = \tilde{m}_{\tilde{\chi}}$  for characters  $\tilde{\chi}$  of  $\tilde{G} = G/N$ ,  $N \triangleleft G$  finite, with  $\tilde{\Gamma} = \Gamma N/N \subset \ker \tilde{\chi}$ .

PROPOSITION 4.2.  $(t_{K/k}|\Gamma)(g^{-1}) \in \Lambda_\wedge \Gamma$  for all  $g \in G \setminus Z(G)$ .

By Proposition 2.3,  $\overline{(t_{K/k}|\Gamma)(g^{-1})} = \frac{1}{[G:\Gamma]} \sum_\chi h_g \chi(g^{-1}) m_\chi \bar{g}^{-1}$ , with the sum ranging over the irreducible  $\chi \in R_l(G)$  which have  $\Gamma$  in their kernel. We need to show

$$\frac{1}{[G:\Gamma]} \sum_\chi h_g \chi(g^{-1}) m_\chi \in \Lambda_\wedge \Gamma_k \text{ for } g \in G \setminus Z(G).$$

If  $g \notin G'$ , then the centralizer of  $g$  in  $G$  is  $Z(G) \cdot \langle g \rangle$ , hence  $h_g = \frac{1}{l} [G : Z(G)]$ . Moreover,  $\chi(g) = 0$  for every non-abelian  $\chi$ , since these are induced from  $G'$ . Therefore, with  $N = [G, G]$ ,

$$\frac{1}{[G:\Gamma]} \sum_\chi h_g \chi(g^{-1}) m_\chi = \frac{1}{l[Z(G):\Gamma]} \sum_{\chi(1)=1} \chi(g^{-1}) m_\chi = \frac{[G:N:\Gamma]}{l[Z(G):\Gamma]} \overline{(t_{K_{ab}/k}|\tilde{\Gamma})(\tilde{g}^{-1})\bar{g}}.$$

This is in  $\Lambda_\wedge \Gamma_k$  since  $N \cap \Gamma = 1$  and  $\frac{[G:N:\Gamma]}{l[Z(G):\Gamma]} = \frac{[G:\Gamma]}{l[Z(G):\Gamma]|N|} = \frac{[G:Z(G)]}{l|N|} \in \mathbb{Z}$ , by 2. of Lemma 4.1 and  $t_{K_{ab}/k} = \mathbb{L}(\lambda_{K_{ab}/k}) \in T(\Lambda_\wedge G^{\text{ab}})$ .

Next, let  $g \in G'$ . Then

$$\frac{1}{[G:\Gamma]} \sum_\chi h_g \chi(g^{-1}) m_\chi = \frac{h_g}{[G:\Gamma]} \left( \sum_{\chi(1)=1} \chi(g^{-1}) m_\chi + \sum_{\chi(1)=l} \chi(g^{-1}) m_\chi \right).$$

If  $\chi(1) = 1$  set  $\chi' = \text{res}_{G'}^G \chi$ . Then  $\chi'$  is trivial on  $[G, G]$ . We denote the sum over all such  $\chi'$  by  $\sum_1$  and obtain

$$\begin{aligned} \sum_{\chi(1)=1} \chi(g^{-1}) m_\chi &= \sum_1 \sum_{i=0}^{l-1} (\chi \omega^i)(g^{-1}) m_{\chi \omega^i} \\ &= \sum_1 \chi'(g^{-1}) \frac{1}{l} \log \prod_{i=0}^{l-1} \frac{L_{K/k}(\chi \omega^i)^l}{\Psi_{L_{K/k}}(\psi_l(\chi \omega^i))} = \sum_1 \chi'(g^{-1}) \frac{1}{l} \log \frac{L_{K/k'}(\chi')^l}{\Psi_{L_{K/k}}(\psi_l(\text{ind}_{G'}^G \chi'))}. \end{aligned}$$

Above, we have used  $\omega(g) = 1$  and  $L_{K/k}(\text{ind}_{G'}^G \chi') = L_{K/k'}(\chi')$ .

If  $\chi(1) = l$ , then  $\chi = \text{ind}_{G'}^G \chi'$  with an abelian character  $\chi'$  of  $G'/\Gamma$  which is non-trivial on  $[G, G]$ . We denote the sum over all such  $\chi'$  by  $\sum_2$  and obtain

$$\begin{aligned} \sum_{\chi(1)=l} \chi(g^{-1}) m_\chi &= \frac{1}{l} \sum_2 (\text{ind}_{G'}^G \chi')(g^{-1}) \frac{1}{l} \log \frac{L_{K/k'}(\chi')^l}{\Psi_{L_{K/k}}(\psi_l(\text{ind}_{G'}^G \chi'))} \\ &= \frac{1}{l} \sum_2 \sum_{i=0}^{l-1} \chi'(g^{-a^i}) \frac{1}{l} \log \frac{L_{K/k'}(\chi'^{a^i})^l}{\Psi_{L_{K/k}}(\psi_l(\text{ind}_{G'}^G \chi'^{a^i}))} = \sum_2 \chi'(g^{-1}) \frac{1}{l} \log \frac{L_{K/k'}(\chi')^l}{\Psi_{L_{K/k}}(\psi_l(\text{ind}_{G'}^G \chi'))}, \end{aligned}$$

where we have used that  $\chi'$  and  $\chi'^{a^i}$  induce the same  $\chi$ .

Collecting everything so far, we see that

$$(\star) \quad \frac{1}{[G:\Gamma]} \sum_\chi h_g \chi(g^{-1}) m_\chi = \frac{h_g}{[G:\Gamma]} \sum_{\chi'} \chi'(g^{-1}) \frac{1}{l} \log \frac{L_{K/k'}(\chi')^l}{\Psi_{L_{K/k}}(\psi_l(\text{ind}_{G'}^G \chi'))}.$$

Multiplying the term in log by  $1 = \frac{\Psi L_{K/k}(\text{ind}_{G'}^G \psi_l \chi')}{\Psi L_{K/k'}(\psi_l \chi')}$  gives

$$(\#) \quad \frac{h_g}{l} \overline{(t_{K/k'}|\Gamma)}(g^{-1})\bar{g} + \frac{h_g}{[G:\Gamma]} \sum_{\chi'} \chi'(g^{-1}) \frac{1}{l} \log \Psi L_{K/k}(\sum_{i=0}^{l-1} (\chi' \circ \text{ver}) \omega^i - l(\chi' \circ \text{ver}))$$

by 1. of Lemma 4.1. Since  $g \notin Z(G)$ ,  $h_g = l$  and  $\overline{(t_{K/k'}|\Gamma)}(g^{-1})\bar{g}$  is integral by  $t_{K/k'} = \mathbb{L}(\lambda_{K/k'}) \in T(\Lambda_\wedge G')$ . With respect to the second summand we observe that multiplying  $\chi'$  by an abelian character  $\theta$  of  $G'/\text{ver}(G^{\text{ab}})$  does not change the character  $\sum_{i=0}^{l-1} (\chi' \circ \text{ver}) \omega^i - l(\chi' \circ \text{ver})$ . This brings the sum  $\sum_{\theta} \theta(g^{-1})$  into the second summand and makes it vanish, if  $g \notin \text{ver}(G^{\text{ab}})$ .

The proof of the proposition is complete since  $\text{ver}(G^{\text{ab}}) \subset Z(G)$ .

COROLLARY.  $\overline{(t_{K/k}|\Gamma)}(g^{-1}) = \frac{h_g}{l} \overline{(t_{K/k'}|\Gamma)}(g^{-1})$  for all  $g \in G'$  which are not in the image of  $\text{ver}$ .

To continue with computing the coefficients  $(t_{K/k}|\Gamma)(z^{-1})$  for  $z \in Z(G)$  we add the assumption  $[G, G] \subset Z(G)$ .

LEMMA 4.3. *If  $[G, G] \subset Z(G)$ , then, with a mod  $G'$  generating the cyclic group  $G/G'$  of order  $l$ ,*

1.  $\forall g' \in G' \exists z \in [G, G] : (g')^{a^i} = g' z^i, z^l = 1$
2.  $\chi'^{\hat{A}} \stackrel{\text{def}}{=} \prod_{i=0}^{l-1} \chi'^{a^i} = \chi'^l$  for all irreducible characters  $\chi'$  of  $G'/\Gamma$
3.  $x \in T' \implies x^{l^v} \in l^v T'.$

Indeed,  $z \stackrel{\text{def}}{=} (g')^{a^{-1}} \in [G, G]$ , so  $z^l = (g')^{a^{l-1}} = 1$ , and  $(g')^{\hat{A}} = g'^l$  implies  $\chi'^{\hat{A}} = \chi'^l$ .

For 3., take a set  $\{b\}$  of representatives in  $G'$  of the orbits of  $G'/\Gamma$  under conjugation by  $A$ . Then  $\{\text{tr}_A(b)\}$  is a  $\Lambda_\wedge \Gamma$ -basis of  $T'$ , so we can write  $x = \sum_b x_b \text{tr}_A(b)$ . We show that  $x^l \in lT'$ : Since  $x^l \equiv \sum_b x_b^l \text{tr}_A(b)^l \pmod{lT'}$ , this follows from  $\text{tr}_A(b)^l \in lT'$ . To see the latter, write  $b^a = bz$ ,  $\hat{z} = 1 + z + \dots + z^{l-1}$ , so  $\text{tr}_A(b)^l = (b\hat{z})^l = b^l \hat{z}^l = l^{l-1} b^l \hat{z} = l^{l-2} (lb^l \hat{z}) \in l^{l-2} T'$  (by  $(b^l)^a = b^l z^l = b^l$ ), as required since  $l$  is odd.

3. now follows by induction on  $v$ .

COROLLARY. *If  $x \in T'$ , then for every  $z \in Z(G)$ ,*

$$\frac{1}{[G':\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log(\text{Det}(1+x)(\chi')) \in \Lambda_\wedge^c \Gamma_k.$$

In fact, as in the remark following Proposition 3.2, we have (with  $x = \frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})} - 1$ )

$$(\diamond) \quad \frac{1}{[G':\Gamma]} \sum_{\chi'} \chi'(z^{-1}) (\text{Det } x)(\chi') \equiv 0 \pmod{l \Lambda_\wedge^c \Gamma_k}.$$

Now, since  $\text{Det}(\sum_b x_b \text{tr}_A(b))(\chi') = \sum_{b,i} \bar{x}_b \bar{b} \chi'(b^{a^i}) = l \sum_b \bar{x}_b \bar{b} + \sum_{b,i} \bar{x}_b \bar{b} (\chi'(b^{a^i}) - 1)$  is divisible

by  $\zeta - 1$  for some  $l$ -power root  $\zeta$  of unity, we see that the logarithmic series  $\log(1 + \text{Det } x)(\chi')$  converges in  $\mathcal{Q}_\wedge \Gamma_k$ , hence

$$\log(\text{Det}(1+x)(\chi')) = \log(1 + (\text{Det } x)(\chi')) = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{1}{\nu} (\text{Det } x^\nu)(\chi').$$

So we obtain

$$\begin{aligned} \frac{1}{[G':\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log(\text{Det}(1+x)(\chi')) &= \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{1}{l\nu} \frac{1}{[G':\Gamma]} \sum_{\chi'} \chi'(z^{-1}) (\text{Det } x^\nu)(\chi') \\ &= \overline{z} \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{1}{l\nu} \overline{(x^\nu|\Gamma)(z^{-1})} \end{aligned}$$

by Proposition 2.3 (on identifying  $\Lambda_\wedge G'$  and  $T(\Lambda_\wedge G')$ ).

Writing  $\nu = r l^\nu$  with  $l \nmid r$ , then the above  $\nu$ -th summand is  $\pm \frac{1}{r l^{\nu+1}} \overline{((x^r)^{l^\nu}|\Gamma)(z^{-1})}$  with  $x^r \in \mathcal{T}'$ , hence  $(x^r)^{l^\nu} \in l^\nu \mathcal{T}'$  by 3. of Lemma 4.3. Thus the corollary follows from  $(\diamond)$ .

PROPOSITION 4.4. *Assume*

1.  $G$  has an abelian subgroup  $G'$  of index  $l$ ,
2.  $[G, G] \subset Z(G)$ ,
3.  $\frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})} \equiv 1 \pmod{\mathcal{T}'}$ .

Then  $t_{K/k} \in T(\Lambda_\wedge G)$  <sup>11</sup>.

Because of Proposition 4.2 the proof only requires checking integrality of the coefficients  $(t_{K/k}|\Gamma)(z^{-1})$  for  $z \in Z(G)$ . Invoking  $(\star)$  and  $(\#)$  of that proposition we see that

$$\begin{aligned} \overline{(t_{K/k}|\Gamma)(z^{-1})z} &= \frac{1}{l} \overline{(t_{K/k'}|\Gamma)(z^{-1})z} + \\ &+ \frac{1}{[G:\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \left( \Psi L_{K/k}(\sum_{i=0}^{l-1} (\chi' \circ \text{ver}) \omega^i - l(\chi' \circ \text{ver})) \right). \end{aligned}$$

By  $\text{ind}_G^{G'}(\text{res}_G^{G'}(\chi' \circ \text{ver})) = \sum_{i=0}^{l-1} (\chi' \circ \text{ver}) \omega^i$  and  $\text{res}_G^{G'}(\chi' \circ \text{ver}) = \chi'^{\hat{A}}$ , the second summand equals  $\frac{1}{[G:\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \Psi \frac{L_{K/k'}(\chi'^{\hat{A}})}{L_{K/k}(\chi' \circ \text{ver})^l}$ . However,

$$\begin{aligned} \frac{1}{[G:\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \left( \Psi \left( \frac{L_{K/k'}(\chi'^{\hat{A}})}{L_{K/k}(\chi' \circ \text{ver})^l} \right) \cdot \frac{L_{K/k'}(\chi')^l}{L_{K/k'}(\chi')^l} \right) \\ = l \cdot \frac{1}{[G:\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \frac{L_{K/k'}(\chi')}{\Psi L_{K/k}(\chi' \circ \text{ver})} + \frac{1}{[G:\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \frac{\Psi L_{K/k'}(\chi'^{\hat{A}})}{L_{K/k'}(\chi')^l}, \end{aligned}$$

and, since  $\psi_l \chi' = \chi'^l$  here, the second summand is

$$\begin{aligned} \frac{1}{[G:\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \left( \frac{\Psi L_{K/k'}(\chi'^{\hat{A}})}{L_{K/k'}(\chi')^l} \cdot \frac{\Psi L_{K/k'}(\psi_l \chi')}{\Psi L_{K/k'}(\chi')^l} \right) \\ = \frac{1}{[G:\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \frac{\Psi L_{K/k'}(\psi_l \chi')}{L_{K/k'}(\chi')^l} + 0 = -\frac{1}{l} \overline{(t_{K/k'}|\Gamma)(z^{-1})z}, \end{aligned}$$

by 2. of Lemma 4.3 and Proposition 2.3.

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<sup>11</sup>and thus  $L_{K/k} \in \text{Det } K_1(\Lambda_\bullet G)$ , by Proposition 3.2

Putting things together, we arrive at

$$\begin{aligned} \overline{(t_{K/k}|\Gamma)(z^{-1})\bar{z}} &= \frac{1}{l} \overline{(t_{K/k'}|\Gamma)(z^{-1})\bar{z}} + \frac{1}{[G':\Gamma]} \sum_{\chi'} \chi'(z^{-1}) \frac{1}{l} \log \frac{L_{K/k'}(\chi')}{\Psi_{L_{K/k}(\chi')\text{over}}} - \\ &- \frac{1}{l} \overline{(t_{K/k'}|\Gamma)(z^{-1})\bar{z}} = \frac{1}{[G':\Gamma]} \sum_{\chi'} (z^{-1}) \frac{1}{l} \log \frac{L_{K/k'}(\chi')}{\Psi_{L_{K/k}(\chi')\text{over}}} . \end{aligned}$$

Now insert condition 3. and the above corollary with  $x = \frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})} - 1$ ; recall that  $\text{Det}(1+x)(\chi') = (\text{Det} \frac{\lambda_{K/k'}}{\text{ver}(\lambda_{K_{\text{ab}}/k})})(\chi') = \frac{L_{K/k'}(\chi')}{\Psi_{L_{K/k}(\chi')\text{over}}}$ . This finishes the proof of the proposition.

## 5. APPENDIX

We begin this section by requiring only that  $S$  contains the infinite primes and those above  $l$ .

**PROPOSITION 5.1.** *The existence of a pseudomeasure in  $K_1(\mathcal{Q}G)$  is independent of the choice of the set  $S$  as above.*

The proof of the proposition starts from the definition of a “ $K_1$ -Euler factor for  $\mathfrak{p}$  in  $K/k$ ” where  $\mathfrak{p}$  is a prime of  $k$  which does not divide  $l\infty$ . In this case, if  $\mathfrak{P}$  is a fixed prime of  $K$  above  $\mathfrak{p}$ , its decomposition group  $D$  is an open subgroup of  $G$  and its ramification subgroup  $I$  a finite normal subgroup in  $D$ . We define

1. if  $\mathfrak{p}$  is undecomposed, i.e.,  $D = G$ ,  $E(\mathfrak{p}, K/k) \stackrel{\text{def}}{=} [\mathcal{Q}G, 1 - \frac{g}{N_{\mathfrak{p}}} \varepsilon] \in K_1(\mathcal{Q}G)$ , where  $\varepsilon = \frac{1}{|I|} \sum_{h \in I} h$ ,  $N_{\mathfrak{p}} = |\mathfrak{o}_k/\mathfrak{p}|$  and  $gI$  is the Frobenius automorphism  $\text{Fr}_{\mathfrak{p}}$  for the unique prime  $\mathfrak{P}$  of  $K$  above  $\mathfrak{p}$ ,
2. in general,  $E(\mathfrak{p}, K/k) \stackrel{\text{def}}{=} \text{ind}_D^G(E(\mathfrak{p}_1, K/k_1)) \in K_1(\mathcal{Q}G)$  with  $\mathfrak{p}_1 = \mathfrak{P} \cap k_1$  and  $k_1 = K^D$ .

Since the primes of  $K$  above  $\mathfrak{p}$  are  $G$ -conjugates of  $\mathfrak{P}$ , this definition does not depend on the choice of  $\mathfrak{P}$ .

**LEMMA 5.2.** *Let  $G'$  be an open subgroup of  $G$  with fixed field  $k'$ . Then  $\text{res}_G^{G'} E(\mathfrak{p}, K/k) = \prod_{\mathfrak{p}'|\mathfrak{p}} E(\mathfrak{p}', K/k')$  with the  $\mathfrak{p}'$  running through all primes of  $k'$  above  $\mathfrak{p}$ .*

**PROOF.** (of the lemma). We first look at the case when  $D = G$  and use coset representatives  $x_j$  ( $1 \leq j \leq e$ ,  $x_1 = 1$ ) of  $I \cap G'$  in  $I$ , so  $I = \bigcup_{j=1}^e x_j(I \cap G')$ , for getting the  $\mathcal{Q}G'$ -basis  $\{g^i \varepsilon, (x_j - 1)g^i : 0 \leq i < f, 1 < j \leq e\}$  of  $\mathcal{Q}G$ . Here,  $e, f$  are the ramification index, respectively residue degree, of  $\mathfrak{p}$  in  $k'/k$ .

Right multiplication by  $1 - \frac{g}{N_{\mathfrak{p}}} \varepsilon$  then yields the  $e \times e$  block matrix, with blocks of size  $f \times f$ ,

$$\begin{pmatrix} M & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} 1 & -\frac{1}{N_{\mathfrak{p}}} & & \\ & \ddots & \ddots & \\ & & 1 & -\frac{1}{N_{\mathfrak{p}}} \\ -\frac{g'}{N_{\mathfrak{p}}} & & & 1 \end{pmatrix}$$



and  $g^f = g'h$  for some  $g' \in G'$  and  $h \in I$ . Of course,  $g' \cdot (I \cap G')$  is the Frobenius automorphism  $\text{Fr}_{\mathfrak{p}'}$  of  $\mathfrak{P}$  in  $K/k'$ . It follows that  $\text{res}_G^{G'}[\mathcal{Q}G, 1 - \frac{g}{N_{\mathfrak{p}}} \varepsilon] \in K_1(\mathcal{Q}G')$  is represented by the above block matrix  $M$  and thus equals (compare [CR, §40C])  $[\mathcal{Q}G', 1 - \frac{g'}{(N_{\mathfrak{p}})^f}] = [\mathcal{Q}G', 1 - \frac{g'}{N_{\mathfrak{p}'}}]$ , as is seen by performing the obvious column operations on  $M$  to arrive at

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ * & \dots & * & 1 - \frac{g'}{(N_{\mathfrak{p}})^f} \end{pmatrix}.$$

The general case of the lemma is reduced to this by means of the Mackey formalism:

The double coset decomposition  $G = \bigcup_{t \in T} G'tD$  yields

$$\text{res}_G^{G'} \text{ind}_D^G = \prod_{t \in T} \text{ind}_{G' \cap D_t}^{G'} \text{res}_{D_t}^{G' \cap D_t} c_t$$

where  $D_t = tDt^{-1}$  and  $c_t : K_1(\mathcal{Q}D) \rightarrow K_1(\mathcal{Q}D_t)$  is induced by  $x \mapsto txt^{-1}$ .

We assume that  $1 \in T$  and start from the fixed prime  $\mathfrak{P}$  in  $K$  above  $\mathfrak{p}$ . Then  $t\mathfrak{P} \cap k'$ , for  $t \in T$ , are precisely the primes of  $k'$  above  $\mathfrak{p}$ . Denote the fixed field of  $D_t$  by  $k_t$  and set  $\mathfrak{p}_t = t\mathfrak{P} \cap k_t$ , so  $\mathfrak{p}_t$  is undecomposed in  $K/k_t$ . Then

$$\begin{aligned} \text{res}_G^{G'} E(\mathfrak{p}, K/k) &= \text{res}_G^{G'} \text{ind}_D^G E(\mathfrak{p}_1, K/k_1) = \prod_t \text{ind}_{G' \cap D_t}^{G'} \text{res}_{D_t}^{G' \cap D_t} c_t E(\mathfrak{p}_1, K/k_1) \\ &= \prod_t \text{ind}_{G' \cap D_t}^{G'} \text{res}_{D_t}^{G' \cap D_t} E(\mathfrak{p}_t, K/k_t) = \prod_t \text{ind}_{G' \cap D_t}^{G'} E(\mathfrak{p}'_t, K/k'_t) \end{aligned}$$

with  $k'_t$  the fixed field of  $G' \cap D_t$  and  $\mathfrak{p}'_t = t\mathfrak{P} \cap k'_t$ , by the first part of the proof. Since  $\mathfrak{p}'_t \cap k' = t\mathfrak{P} \cap k'$  divides  $\mathfrak{p}$  and since the decomposition group of  $t\mathfrak{P}$  in  $K/k'$  is  $G' \cap D_t$ , the above product is  $\prod_t E(t\mathfrak{P} \cap k', K/k')$ , by the definition of the  $K_1$ -Euler factor in the general case.

This finishes the proof of the lemma and we turn back to the proof of Proposition 5.1 for which it now suffices to show that  $\mathfrak{p} \notin S$  implies

$$\text{Det } E(\mathfrak{p}, K/k) = \frac{L_{K/k, S \cup \{\mathfrak{p}\}}}{L_{K/k, S}}.$$

We check this by evaluating both sides of the claimed equality at the characters  $\chi$  of  $G$ . By Brauer induction we may assume that  $\chi = \text{ind}_{G'}^G \chi'$  is induced from a degree 1 character of some open subgroup  $G'$  of  $G$ . Now, with an obvious notation, Lemma 5.2 gives

$$\begin{aligned} (\text{Det } E(\mathfrak{p}, K/k))(\chi) &= (\text{Det}(\text{res}_G^{G'} E(\mathfrak{p}, K/k)))(\chi') = \prod_{\mathfrak{p}'|\mathfrak{p}} (\text{Det } E(\mathfrak{p}', K/k'))(\chi') \\ &= \prod_{\mathfrak{p}'|\mathfrak{p}} \text{Det}[\mathcal{Q}G', 1 - \frac{g_{\mathfrak{p}'}}{N_{\mathfrak{p}'}} \varepsilon_{\mathfrak{p}'}](\chi'). \end{aligned}$$

Since  $1 - \frac{g_{\mathfrak{p}'}}{N_{\mathfrak{p}'}} \varepsilon_{\mathfrak{p}'} = 1 - \frac{1}{N_{\mathfrak{p}'}} \cdot \frac{1}{|I_{\mathfrak{p}'}|} \sum_{h \in I_{\mathfrak{p}'}} g_{\mathfrak{p}'} h$ , the value of  $\text{Det}[\mathcal{Q}G', 1 - \frac{g_{\mathfrak{p}'}}{N_{\mathfrak{p}'}} \varepsilon_{\mathfrak{p}'}]$  at the 1-dimensional  $\chi'$  is  $= 1 - \frac{1}{N_{\mathfrak{p}'}} \chi'(g_{\mathfrak{p}'}) \chi'(\varepsilon_{\mathfrak{p}'}) \bar{g}_{\mathfrak{p}'}$  (compare the first two displayed formulas

following FACT 1 in §1 and note that in the 1-dimensional case Det and Tr coincide). The above expression equals  $1 - \frac{\chi'(g_{\mathfrak{p}'})}{N\mathfrak{p}'} \bar{g}_{\mathfrak{p}'}$  or 1 according as  $I_{\mathfrak{p}'} \subset \ker \chi'$  or not. Thus

$$(\text{Det } E(\mathfrak{p}, K/k))(\chi) = \prod_{\mathfrak{p}'|\mathfrak{p}, I_{\mathfrak{p}'} \subset \ker \chi'} \left(1 - \frac{\chi'(\text{Fr}_{\mathfrak{p}'})}{N\mathfrak{p}'} \overline{\text{Fr}_{\mathfrak{p}'}}\right)$$

and we are left with showing that this product is also

$$= \frac{L_{K/k, S \cup \{\mathfrak{p}\}}(\chi)}{L_{K/k, S}(\chi)} = \frac{L_{K/k', S' \cup \{\mathfrak{p}'\}}(\chi')}{L_{K/k', S'}(\chi')}$$

with  $\{\mathfrak{p}'\}$  again the set of primes of  $k'$  above  $\mathfrak{p}$ .

Now, with  $u' \in 1 + l\mathbb{Z}_l$  denoting the action of a generator  $\gamma_{k'}$  of  $\Gamma_{k'}$  on  $l$ -power roots of unity, the ratio of power series  $\frac{G_{\chi', S' \cup \{\mathfrak{p}'\}}(T)}{G_{\chi', S'}(T)}$  is uniquely determined by its values

$$\begin{aligned} \frac{G_{\chi', S' \cup \{\mathfrak{p}'\}}((u')^{n-1})}{G_{\chi', S'}((u')^{n-1})} &= \frac{L_{l, S' \cup \{\mathfrak{p}'\}}(1-n, \chi')}{L_{l, S'}(1-n, \chi')} = \prod_{\mathfrak{p}'|\mathfrak{p}, I_{\mathfrak{p}'} \subset \ker \chi'} \left(1 - \frac{\chi'(\text{Fr}_{\mathfrak{p}'})}{N\mathfrak{p}'} (N\mathfrak{p}')^n\right) \\ &= \prod \left(1 - \frac{\chi'(\text{Fr}_{\mathfrak{p}'})}{N\mathfrak{p}'} \langle N\mathfrak{p}' \rangle^n\right) = \prod \left(1 - \frac{\chi'(\text{Fr}_{\mathfrak{p}'})}{N\mathfrak{p}'} (u')^{nb_{\mathfrak{p}'}}\right) \end{aligned}$$

at all natural numbers  $n \equiv 0 \pmod{l-1}$ . Here,  $\langle N\mathfrak{p}' \rangle \in 1 + l\mathbb{Z}_l$  is determined by  $(N\mathfrak{p}')^n = \langle N\mathfrak{p}' \rangle^n$  for all such  $n$  and  $b_{\mathfrak{p}'} \in \mathbb{Z}_l$  by  $\langle N\mathfrak{p}' \rangle = (u')^{b_{\mathfrak{p}'}}$ . Consequently, substituting  $T = \gamma_{k'} - 1$ ,

$$\frac{L_{K/k', S' \cup \{\mathfrak{p}'\}}(\chi')}{L_{K/k', S'}(\chi')} = \frac{G_{\chi', S' \cup \{\mathfrak{p}'\}}(\gamma_{k'} - 1)}{G_{\chi', S'}(\gamma_{k'} - 1)} = \prod \left(1 - \frac{\chi(\text{Fr}_{\mathfrak{p}'})}{N\mathfrak{p}'} \gamma_{k'}^{b_{\mathfrak{p}'}}\right)$$

and so  $\overline{\text{Fr}_{\mathfrak{p}'}} = \gamma_{k'}^{b_{\mathfrak{p}'}}$  finishes the proof of the proposition: these automorphisms act on  $l$ -power roots of unity by  $\langle N\mathfrak{p}' \rangle = (u')^{b_{\mathfrak{p}'}}$ .

We close this section by recalling the “equivariant main conjecture”. The set  $S$  is now supposed to be sufficiently large. Moreover, we assume  $\mu = 0$  for  $K/k$  <sup>12</sup>.

Let  $M$  be the maximal abelian  $l$ -extension of  $K$ , which is unramified outside  $S$ , and set  $X = G_{M/K}$ . Then  $X$  is a finitely generated torsion  $\Lambda G$ -module <sup>13</sup> (the so-called Iwasawa module). Though  $X$  generally does not have finite projective dimension itself, it naturally induces an element <sup>14</sup>  $\mathfrak{U}$  in the Grothendieck group  $K_0 T(\Lambda G)$  of all finitely generated torsion  $\Lambda G$ -modules with finite projective dimension. This  $\mathfrak{U}$  not only keeps all the information of the Iwasawa module  $X$ , but also includes extension class data. For its derivation see [RWt or RW2].

The localization sequence of  $K$ -theory provides a connecting homomorphism

$$K_1(\mathcal{Q}G) \xrightarrow{\partial} K_0 T(\Lambda G)$$

<sup>12</sup>the assumption is independent of  $S$  as long as  $l \in S$ , see [NSW, (11.3.6), p.615]

<sup>13</sup>As before, “torsion” means that there is a central regular  $c \neq 0$  in  $\Lambda G$  which annihilates  $X$ .

<sup>14</sup>We prefer to just write  $\mathfrak{U}$  and  $\Theta$  rather than  $\tilde{\mathfrak{U}}, \tilde{\Theta}$ , as in [RW 2,3,4]. Note however that these  $\mathfrak{U}, \Theta$  differ from the ones appearing in [RWt, RW1]: compare [RW2, Remark A., p.564].

and  $\partial$  has  $\mathfrak{U}$  in its image [RW2, Lemma 13]. Combining  $\partial$  and Det yields

$$\begin{array}{c} K_1(\mathcal{Q}G) \xrightarrow{\partial} K_0T(\Lambda G) \\ \text{Det} \downarrow \\ \text{Hom}^*(R_l(G), (\mathcal{Q}^c\Gamma_k)^\times) \end{array}$$

and the question arises whether there is a common source in  $K_1(\mathcal{Q}G)$  for the two distinguished elements,  $\mathfrak{U}$  at the right and  $L_{K/k}$  at the bottom. The “equivariant main conjecture” of Iwasawa theory, as stated in [RW2], asserts that, roughly speaking, there is a preimage  $\Theta$  of  $L_{K/k}$  in  $K_1(\mathcal{Q}G)$  such that  $\partial(\Theta) = \mathfrak{U}$ . If  $G$  is abelian, this is essentially what the Main Conjecture of classical Iwasawa theory is about (compare [RW1]). In [RW3,4] it has been shown that, if  $\mu = 0$ , the existence of  $\Theta$  is equivalent to  $L_{K/k}$  belonging to  $\text{Det } K_1(\Lambda_\bullet G)$ . Then  $\Theta$  is a non-abelian pseudomeasure for  $K/k$ .

PROPOSITION 5.3. *The “equivariant main conjecture” does not depend on the choice of a sufficiently large set  $S$ .*

It is enough to show

$$(*) \quad \partial E(\mathfrak{p}, K/k) = \mathfrak{U}_{S \cup \{\mathfrak{p}\}} - \mathfrak{U}_S$$

for  $\mathfrak{p} \notin S$ . In the notation of the general case of the definition of a  $K_1$ -Euler factor we have

$$E(\mathfrak{p}, K/k) = \text{ind}_D^G E(\mathfrak{p}_1, K/k_1) = \text{ind}_D^G [\mathcal{Q}D, 1 - \frac{g_1}{N_{\mathfrak{p}_1}} \varepsilon_1] = \text{ind}_D^G [\mathcal{Q}D\varepsilon_1, 1 - \frac{g_1}{N_{\mathfrak{p}_1}}]$$

because  $1 - \frac{g_1}{N_{\mathfrak{p}_1}} \varepsilon_1$  acts on  $\mathcal{Q}D = (\mathcal{Q}D)\varepsilon_1 \oplus (\mathcal{Q}D)(1 - \varepsilon_1)$  as  $(1 - \frac{g_1}{N_{\mathfrak{p}_1}})\varepsilon_1 + (1 - \varepsilon_1)$ . Since  $(\mathcal{Q}D)\varepsilon_1 = \mathcal{Q}(D/I)$ ,  $N_{\mathfrak{p}_1} = N_{\mathfrak{p}}$ , and as  $g_1$  acts here as  $\text{Fr}_{\mathfrak{p}_1}$  this becomes

$$\partial E(\mathfrak{p}, K/k) = \partial \left( \text{ind}_D^G [\mathcal{Q}(D/I), 1 - \frac{\text{Fr}_{\mathfrak{p}_1}}{N_{\mathfrak{p}}}] \right).$$

To compare this with the right hand side of  $(*)$  we now apply the proof of [RWt, Proposition 4.7] (which is not built on Leopoldt’s conjecture). Because  $S$  is sufficiently large, this only requires us to restate the second and fourth displayed formula on [loc. cit., p.38] as

$$\mathfrak{U}_{S \cup \{\mathfrak{p}\}} - \mathfrak{U}_S = [\text{ind}_D^G C] = \partial \left( \text{ind}_D^G [\mathcal{Q}(D/I), 1 - \frac{\text{Fr}_{\mathfrak{p}_1}}{N_{\mathfrak{p}}}] \right).$$

Note that in [RWt] we inverted only the “never zero divisors”  $R \subset \Lambda G$ , so these assertions remain true on inverting the larger set of regular central elements.

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